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TIME-CONSUMING PRODUCTION AND ECONOMIC GROWTH

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ABSTRACT

The paper examines theoretical implications of time-consuming production on economic growth based on technological change. Inspired by Austrian theory of capital we build a model of technological change in which various stages of production are formalized as a sequence of CES-production functions. We assume one R&D sector. Successful research consists in invention of new stages of production. Short-run equilibriums when number of stages of production is constant as well as long-run equilibriums are solved. We conclude that endogenous growth is not possible within a simple framework. Modifications of the model allowing for endogenous growth, namely *production-accelerating* technological progress, are proposed.

KEYWORDS: economic growth, stages of production, structure of production, technological change, time structure of capital

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INTRODUCTION

Robert M. Solow in his revolutionary article Solow (1956) has shown how diminishing marginal product of capital constitutes barrier to long-run economic growth. Since then, a lot of attention has been paid to mechanism of overcoming diminishing marginal product of capital and produced inputs in general. Three important approaches have been developed: The first one introduces spill-over effects and so called learning-by-doing (e.g., Romer, 1986). In these models, positive externalities in the form of spill-over offset diminishing marginal product. The second one starting with Lucas (1988) deals with a broad notion of capital including *human capital* which unlike *raw labour* can be accumulated and thus helps to overcome tendency to diminishing marginal product. The last one introduces substitutable *producer goods*, increasing variety or quality of which enable constant returns to scale and an endogenous growth (Romer, 1987 and 1990, Grossman and Helpman, 1991, Aghion and Howitt, 1992); we refer to these models as models of technological change.

Models based on these assumptions have one common characteristic: no time is needed to transform inputs into output – production is instantaneous. Even in the case of models of technological change with increasing variety of producer goods which are not perfectly substitutable in production (first introduced by Romer, 1990), there is no difference in the way they are produced or used as inputs. Assumption of perfect homogeneity has been only partially relaxed by Grossman and Helpman (1991) and Aghion and Howitt (1992) who introduced quality ladders and allowed for producer goods of different quality. Both approaches (one based on increasing variety and one based on increasing quality of producer goods) retained an assumption of instantaneous production; various producer goods, whether of different quality or not, are combined into final good at the single moment in time.

Kydland and Prescott (1982) introduced time factor in the analysis of business cycle. In their model, multiple periods are required to build new capital and only finished capital enters production process. They called this time-to-build technology and showed its importance in economic fluctuations.

Asea and Zak (1999) introduced time-to-build technology in a simple model of neoclassical growth as formulated by Cass (1965) and Koopmans (1965). They focused on transitional dynamics to steady state and showed that lag between investment and production leads to oscillatory behavior of the model. Winker et al. (2005) investigated time-to-build factor in neoclassical model with Leontief production function and also focused on transitional dynamics, mainly on the relationship between the frequencies of oscillations and the length of time lag between investment and production. Bambi (2006) worked with time-to-build factor in the AK-model and his emphasis is also oscillatory transitional dynamics to the steady state.

Our approach differs from above mentioned publications in several aspects. Firstly, inspired by Austrian theory of capital, we assume that not only production of durable capital requires multiple periods, but we generalize this feature to all production. We refer to this technology as *time-consuming* production. Secondly, we assume that number of periods needed to complete production process changes as technology improves. Thirdly, we use neither neoclassical, nor AK model as our working framework; instead, we incorporate time factor in the model of technological change. Lastly, we are not focused on the short-run transitional dynamics; our emphasis is on the possibilities of long-run economic growth. We will argue that an introduction of time factor may significantly alter the way we think about growth and about technological change especially. Furthermore, this paper can be understood as an attempt to formalize the main ideas of Austrian theory of capital and to discuss its role in the theory of economic growth.

Following section explains our motivation to analyze above mentioned issues and gives an overview of Austrian theory of capital. In Section 2 we set up the model, in Section 3 we solve for short-run and long-run steady states. As it will turn out that the model is not compatible with endogenous growth, in Section 4 we propose various ways of modifying the model to allow for endogenous growth. Final section concludes the paper.

1. MOTIVATION AND THEORETICAL BACKGROUD

Since the year 1500, world GDP per capita has grown by the factor of 11.5.[2] In some regions, increase in material well-being was even higher, today's product per capita in Western Europe being 28 times the value in 1500, in North America 75 times the value in 1500. However, nor average North American consumes 75 times more *kilograms* of goods, nor average Western European 28 times more. The main difference between today's production and production in 1500 is not in its quantity, but in its *quality.* Our ability, our know-how to produce out of the same inputs (labour and soil) products and services which yield more utility, this is arguably the essence of technological progress, technological change and ultimate source of economic growth.[3]

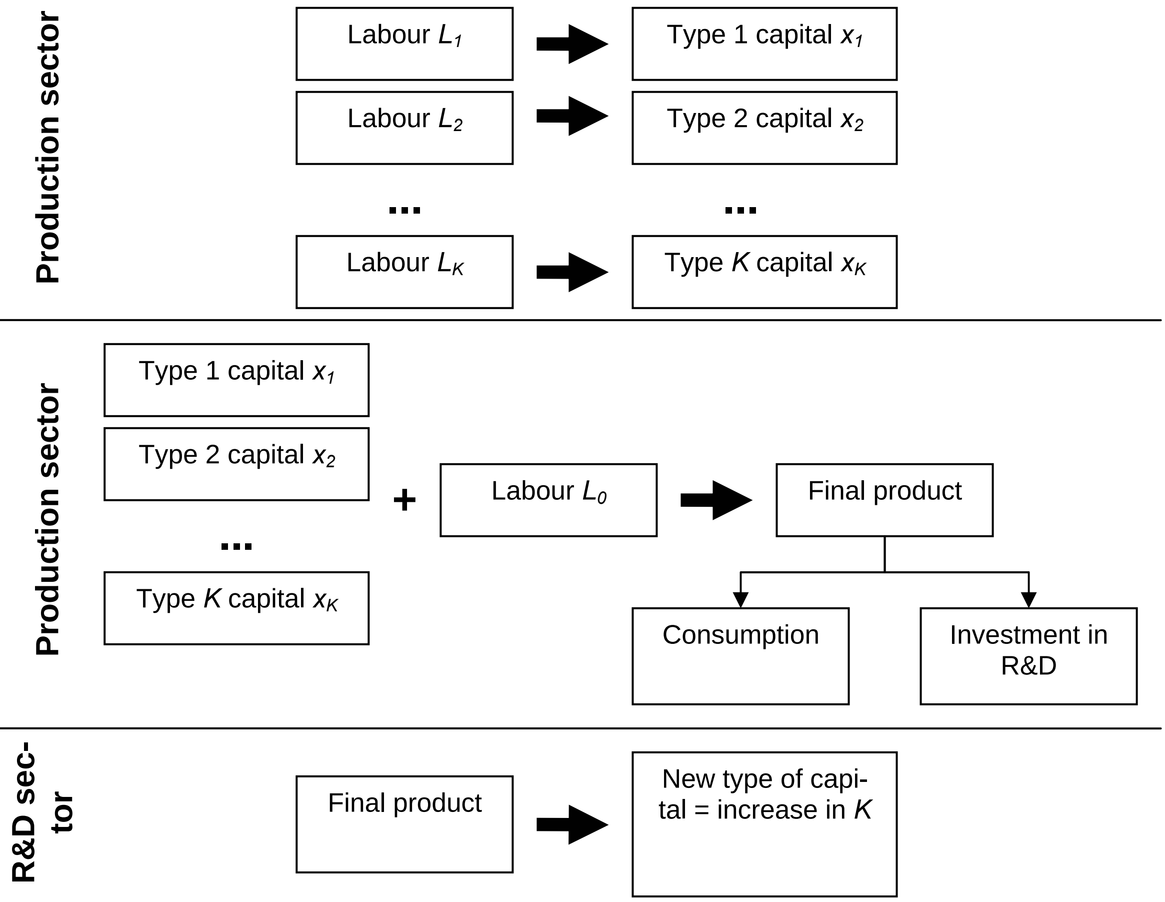
This idea was formalized by Romer (1990) who in his pioneering work focused on increasing number of 'designs,' each design being the set of instruction for mixing together raw materials. Larry E. Jones and Rodolfo E. Manuelli in their overview of theories of economic growth *The Sources of Growth* build similar model based on following assumptions: At any given time there exists K designs according to which, in K different sectors, K types of perishable capital goods are produced out of labour. These K types of capital are *instantaneously* mixed with labour to produce final product. One part of final product is consumed; another part is used as input in R&D where new designs are invented. Crucial assumption is that increase

[2] All historical data are according to Maddison (2010).

[3] „*The raw materials that we use have not changed, but as a result of trial and error, experimentation, refinement, and scientific investigation, the instructions that we follow for combining raw materials have become vastly more sophisticated. One hundred years ago, all we could do to get visual stimulation from iron oxide was to use it as a pigment. Now we put it on plastic tape and use it to make videocassette recordings.*" Romer (1990), p. 72.

in one type of capital goods does not diminish marginal product of another types of capital goods. Therefore, increasing number of designs leads to long-run economic growth. Functioning of the Jones-Manuelli is depicted in Scheme 1. From the formal point of view, a model which we present in this paper can be understood as direct modification of Jones-Manuelli model.

Scheme 1
Instantaneous production in Jones-Manuelli model



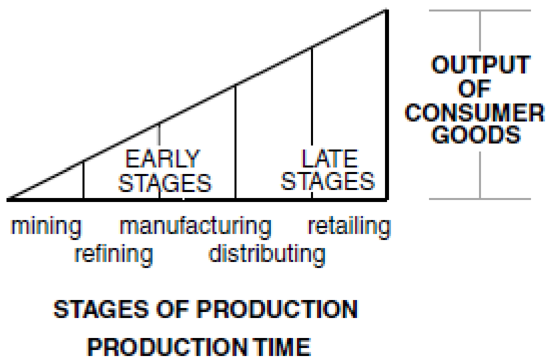
However, there is one feature of production process that can be considered missing in Romer's or in Jones-Manuelli model. To 'assemble' K different types of capital into one final product takes time. Much more different operations are needed to create more sophisticated final product. Think about medieval scholars using an abacus. To construct the abacus craftsmen chop a tree, manufacture a wooden frame, bars and balls, drill a hole in every ball and assemble the abacus. Now suppose that scholars find a textbook thoroughly explaining how to construct a modern calculator. They hire the best craftsmen to do it. However, innumerably more steps are necessary to construct the calculator as opposed to the abacus. Different ores and raw materials are to be mined and finely processed. Tiny semiconductor and plastic components are to be manufactured out of which complicated integrated circuits and digital display are to be constructed. Battery is to be made out of conductive electrolyte and electrodes and so on. It would not only take more *steps* to construct a calculator, it would take more *time* to do that. In the medieval state of technology, it might take years, maybe decades to construct a calculator.

The emphasis on time element in production process can be traced back to Menger (1871), Jevons (1871) and Bőhm-Bowerk (1884) and it constitutes the backbone of Austrian theory of capital (ATC).[4] In the following brief exposition of ATC we follow Roger W. Garrison's *Time and Money*.

According to ATC, it is useful to abandon the assumption of instantaneous production in favour of time-consuming production. Production process is presented as a sequence of stages of production (e.g. mining, refining, manufacturing, distributing and retailing). Output from one stage is used as an input in subsequent stage. Such a linear process is most readily imagined on the basis of goods-in-process.[5]

Hayekian triangle (see Figure 1) first introduced by Hayek (1935) is the most popular representation of the main idea of ATC.

F i g u r e 1
The structure of production



Source: Garrison, 2001.

Hayekian triangle represents a process where original factors of production (i.e. labour and soil) are gradually transformed into consumer goods. Expenditures on consumption appear on vertical leg of the triangle. On the horizontal leg, time appears. Unfinished goods move from early stages to late stages, in each stage certain value being added by original factors

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[4] Mises (1912, 1966) was the first to use this approach to analyse business cycle. Friedrich A. von Hayek gave profound analysis of business cycles in Hayek (1935) and elaborated the theory in Hayek (1941). Garrison (2001) attempts to integrate ATC into broad modern macroeconomic context. It has to be noted that economists of Austrian school of thought are in general reluctant to mathematical formalization of these ideas. The only paper known to us which attempts to formalize ATC is early Thompson (1936).

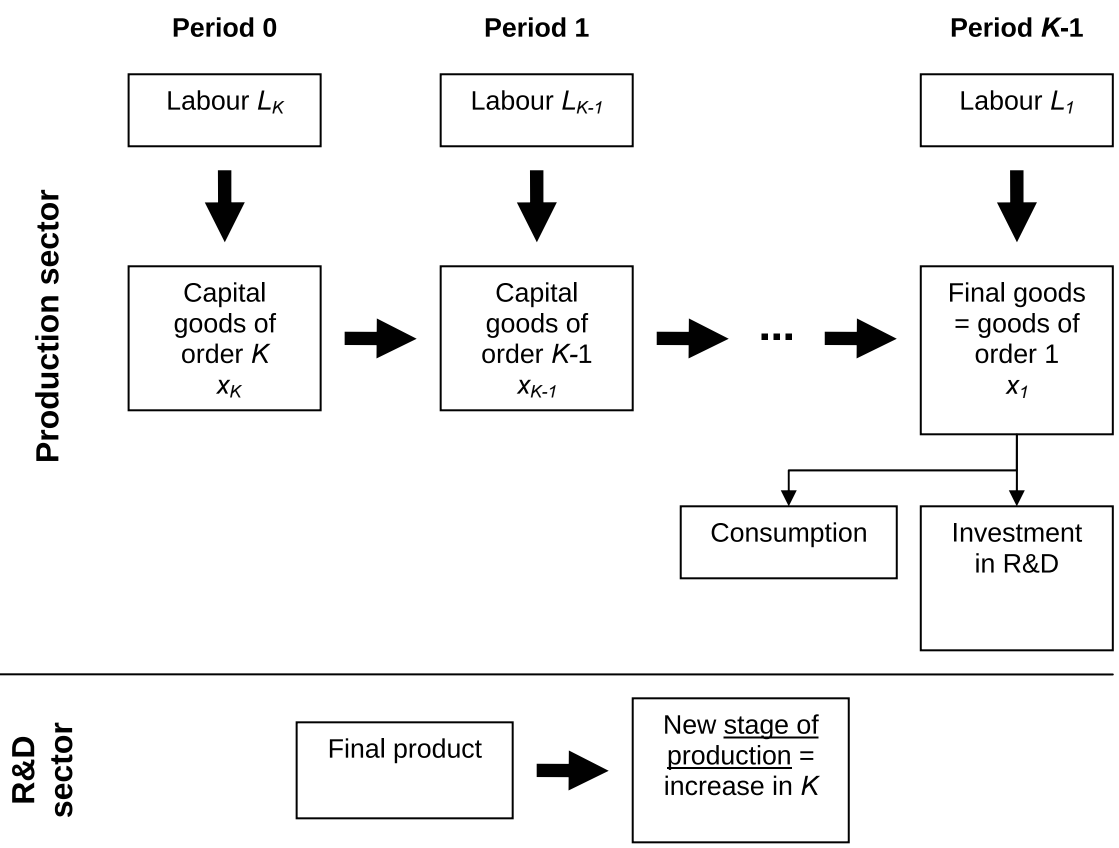
[5] However, as Hayek and Garrison acknowledge, *"there are many feedback loops, multiple-purpose outputs, and other instances of nonlinearities. Further, each stage may also involve the use of durable – but depreciating – capital goods, relatively specific and relatively non-specific capital goods, and capital goods that are related with various degrees of substitutability and complementarity to the capital goods in other stages of production."* Garrison (2001), p. 25-26.

of production employed in given stage (most importantly labour). Capital goods produced by early stages called *goods of high orders* are turned into *goods of low orders* and finally into consumer goods. Alternatively, in Figure 1, height of the each stage's 'column' represents value of current stock of capital goods of given order.

We propose to formalize these ideas in a way similar to Jones-Manuelli model. Think about the economy depicted in Scheme 2. As opposed to Jones-Manuelli model, final product is produced gradually. Product 'moves' from one stage of production to another, workers in each stage add some value to the product and it takes exactly one period to do that. As in Jones-Manuelli model, final output can be used either for consumption or as an input in research and development – successful research increases the number of stages of production.

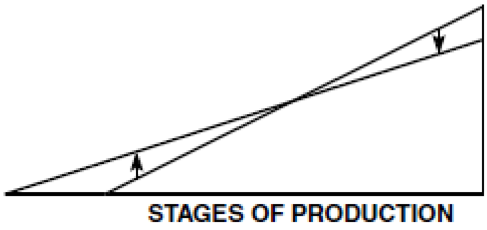
In Jones-Manuelli model labour L_1 is used to produce capital good x_1, labour L_2 to produce capital good x_2 and labour L_3 to produce capital good x_3. Then, x_1, x_2 and x_3 are mixed with labour L_0 to produce final product. In our model, labour L_3 is used to produce capital good x_3. In the next period, labour L_2 is applied to x_3 to turn it to x_2. Similarly, in the next period, out of x_2 and L_1 capital good x_1 is produced *which is final product* (in the following text we use Y's instead of x's). There is no counterpart to L_0 in our model.

Scheme 2
Time-consuming production



As for the general predictions of the model depicted in Scheme 2, we pick two issues from ATC: (1) impact of the time preference and (2) so called ‘secular growth’.

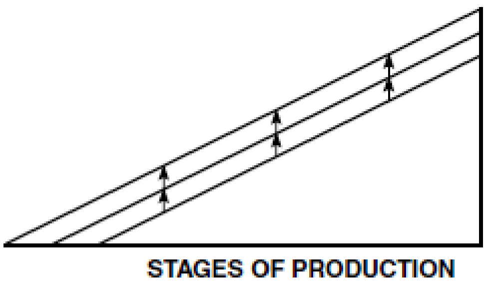
F i g u r e 2
Decrease in time preference



Source: Garrison, 2001.

Within the framework of Hayekian triangle, the time preference determines the slope of the hypotenuse. The slope moves in the same direction as the time preference (see Figure 2). According to ATC, decrease in time preference allows more stages of production to be employed. In other words, number of stages of production in use depends not only on current technology but also on time preference. We will argue against this claim. On the other hand, our analysis will support the claim that decrease in time preference leads to tendency for more labour being employed in early stages of production.

F i g u r e 3
Secular growth



Source: Garrison, 2001.

So called secular growth is depicted in Figure 3. Garrison define secular growth as follows: *“Secular growth occurs without having been provoked by policy or by technological advance or by a change in intertemporal preferences. Rather, the ongoing gross investment is sufficient for both capital maintenance and capital accumulation.”*[6] Thus, economic growth driven solely by the accumulation of capital goods is depicted as an outward shift

[6] Garrison (2001), p. 54.

of the hypotenuse. As the economy grows, more stages of production are used and more consumer goods are produced while the slope of the hypotenuse – in pure market economy determined solely by the time preference – remains unchanged. In this paper we will argue against the possibility of this type of growth if Hayekian triangle is interpreted in terms goods-in-process.[7]

In following section we propose formalization of production process based on series of CES-production functions. Furthermore, analogically to Jones-Manuelli model and Romer's model, we assume that the number of stages of production cannot be expanded at will, but is the result of R&D.

We will abstract from all market imperfections. However, even if production of producer and consumer goods is conformable with assumption of perfect competition, investment R&D is not. As we do not want to enter the discussion of how our model economy behaves under different assumptions about the market structure, we solve the model as *social-planner* problem. Think about the model as describing behaviour of an economy under *optimal policy*.

2. SET-UP OF THE MODEL

Since we want the production to take exactly one period, it is convenient to set a model in discrete time. In period t, technology is such that there are K_t stages of production – there are goods of K_t different orders. To produce goods of i-th order (i being less than K_t), labour and goods of order $i+1$ are needed. However, only labour is needed to produce goods of order K_t. As one period is necessary to transform goods of i-th order into goods of order i-1, goods of i-th order produced in period t enter into production of goods of order i-1 only in period t+1. We will refer to sector producing goods of order i as *i-th order sector* or, alternatively, *i-th stage of production.* We assume diminishing marginal products of both factors of production and constant returns to scale. Production function for i-th order sector can be written in the following form:

$$Y_{i,t} = f\left(Y_{i+1,t-1}, L_{i,t}\right) \text{ for } i < K_t;\, f_{Y_{i+1,t-1}}(\bullet) > 0;\, f_{L_{i,t}}(\bullet) > 0;\, f^2{}_{Y_{i+1,t-1}}(\bullet) < 0;\, f^2{}_{L_{i,t}}(\bullet) < 0$$

$$f\left(\lambda Y_{i+1,t-1}, \lambda L_{i,t}\right) = \lambda Y_{i,t} \text{ for all } \lambda > 0$$

$$Y_{i,t} = \phi\left(L_{i,t}\right) \text{ for } i = K_t;\; \phi' = 1$$

[7] In addition to the accumulation of capital goods, Garrison discuss impacts of changes in technology, resource availabilities and time preferences on economic growth. Changes in technology and resource availabilities are in general modelled in the same way as secular growth i.e. as outward shifts of the hypotenuse. Decrease in time preference, according to Garrison, allows for faster accumulation of capital goods, faster shifting of hypotenuse outward and thus for higher rate of economic growth. Unlike in neoclassical model, changes in time preference do not have 'level effects', on the contrary, conclusions of ATC are similar to those of AK-model of economic growth: changes in time preference have 'growth effects.'

Y_i – goods of i-th order; L_i – labour allocated in the production of goods of i-th order; K_t – number of stages of production in period t

We specify production function as CES-production function:

$$
\begin{aligned}
&Y_{i,t} = \left(Y_{i+1,t-1}{}^{\alpha} + L_{i,t}{}^{\alpha}\right)^{\frac{1}{\alpha}} \text{ for } i < K_t \\
&Y_{i,t} = L_{i,t} \text{ for } i = K_t \\
&\sigma \equiv \frac{1}{1-\alpha};\ -\infty > \alpha > 1
\end{aligned}
\tag{1}
$$

α – coefficient; σ – elasticity of substitution

In each stage, production is performed by perfectly competitive firms. Total endowment of labour is given and is equal to 1. Since leisure (or labour) will not enter utility function, total labour supply is always equal to 1.

One part of the total output is used for consumption; another part is used as an input in R&D. More input in R&D leads to higher probability of successful research, i.e. higher probability of inventing new stage of production. As inputs in R&D approaches infinity, probability of success converges to maximum probability level *B*:

$$
\begin{aligned}
&P\left(K_{t+1} = K_t + 1 \mid R_t\right) = B\left[1 - e^{-AR_t}\right] \\
&0 < B \le 1; A > 0
\end{aligned}
\tag{2}
$$

R_t – expenditures on R&D; B – maximum probability; A – efficiency coefficient

We do not specify R&D sector any further. In our model, optimal amount of resources will be allocated in R&D in every period. Think about R&D as run by government in the best possible way. Since in reality amount of resources devoted to R&D depends heavily on market structure, behaviour of the model would slightly change once details about R&D are specified.[8] However, it is not probable that general conclusions of the model will be changed.[9]

[8] Several points can be made here: (1) R&D can not be run by perfectly competitive firms producing producer and consumer goods in production sector. Since profits in production sector are always zero, there is not enough incentive to perform R&D. (2) R&D can be run by competitive firms if we assume that in the case of successful research, firm may retain monopoly in newly invented stage of production (temporarily on permanently). In this case, behaviour of economy would depend on the market structure in every single stage of production. If we allow for the possibility that the monopoly in *i*-th order sector can be retained even after a sector of order *i*+1 is invented, the market structure in the economy as a whole would be a mixture of perfect competition, monopoly, monopsony and bilateral monopoly. In general, static underproduction will be the result. Impact on R&D expenditures is hard to predict because of complicated structure of markets. (3) We may also assume that once new stage of production is invented, all firms expect successful inventor have to pay sunk costs to enter the sector. If these costs are less then monopoly profits in new stage of production, firms are motivated to enter the sector. This leads to limit pricing from the part of the inventor in new sector. Monopolistic competition in all sectors will lead to less final product. There will be a tendency for R&D expenditures to be less then optimal; however, actual R&D expenditures would be influenced by the extent of *business-stealing effects* (i.e. shifting of profits from early invented stages of production to newly invented stages of production). In the case of significant business-stealing effects, overinvestment in R&D might be the result. We consider the third approach to be the most promising.

We assume rational agents whose preferences are given by following utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t)$$
$$u(C_t) = \ln(C_t) \tag{3}$$
$$\beta \equiv \frac{1}{1+\rho};\ \rho > 0$$

C – consumption; β – discount factor; ρ - coefficient of time preference; t – time

3. SOLVING THE MODEL

There are two distinct optimization problems to be solved in our model:

The first one consists in finding an optimal allocation of labour between different stages of production, their number taken as given. As the probability of successful research is typically low, number of stages of production – K_t – can be treated as constant in the short run i.e. $K_t = K$. This is why we refer to this problem as *short-run optimization problem.* Once the optimal allocation is achieved, economy is in the *short-run equilibrium.* If allocations of labour between various sectors and volumes of producer goods of every order are such that it is optimal to keep them constant, economy is in the *short-run steady state.*

The second problem consists in finding an optimal allocation of total final product between consumption and R&D. If higher number of stages of production enables to produce more output, it might be optimal to devote part of the final output to R&D. We refer to this problem as *long-run optimization problem.* As the economy is most of the time in short-run steady state characteristics of which are given only by the number of stages of production, ratio of total output devoted to R&D can be treated as determined solely by the number of stages of production and also taken as constant in the short-run.

3.1. Finding a short-run equilibrium

In the short-run equilibrium, such allocations of total labour supply between different sectors are chosen that total utility given by (3) is maximized under the constraints given by production function and total labour supply being equal to 1 and with respect to initial endowments

[9] In the following text we will assume that a new stage of production is always stage of order K+1, i.e. if research is successful, new stage of production always appear at the beginning of production process. Since production always starts by extraction of raw materials and ends by retailing, it would be more reasonable to assume that new stages of production appears somewhere in the middle of production process. However, as we do not model transition from one state of technology (with K stages of production) to another state of technology (with K+1 stages of production) and we focus mainly on steady states, this is only matter of notation.

of producer goods. It is convenient to think about initial endowments as of goods produced in the period -1.

Production function (1) can be rewritten as follows:

$$Y_{1,t}=\left(Y_{2,t-1}{}^{\alpha}+L_{1,t}{}^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$Y_{2,t}=\left(Y_{3,t-1}{}^{\alpha}+L_{2,t}{}^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$\dots$$
$$Y_{K-1,t}=\left(Y_{K,t-1}{}^{\alpha}+L_{K-1,t}{}^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$Y_{K,t}=L_{K,t}$$

Inserting $Y_{K,t}$ into production function of $Y_{K-1,t}$, then inserting $Y_{K-1,t}$ into production function of $Y_{K-2,t}$ and so on simplifies production function:

$$Y_{1,t}=\left(L_{1,t}{}^{\alpha}+L_{2,t-1}{}^{\alpha}+\dots+L_{K,t-K+1}{}^{\alpha}\right)^{\frac{1}{\alpha}} \tag{4}$$

We will refer to the production function (4) as *short-run production function.*[10]

[10] Now it is clear why our model can be understand as a modification of model of Jones and Manuelli (1997), section 6. Jones and Manuelli use production function that can be written as (we changed the notation of Jones and Manuelli to correspond to the one we use)

$$Y_t=DL_{1,t}{}^{1-\mu}\left[\int_0^K\left[x(i)^{\alpha}\right]di\right]^{\frac{\mu}{\alpha}}$$

where Y_t denotes final product, D is a constant and $L_{1,t}$ denotes labour allocated to production of final goods. There is a continuum of intermediate goods x(i), $i\in\left[0,K_t\right]$, K_t being their number. Intermediate goods are produced according to production function $x_t(i)=aL_{2,t}(i)$ where a is a constant and $L_{2,t}(i)$ denotes amount of labour used in production of intermediate good i. After inserting production function of intermediate goods into production function of final goods we obtain production function similar to (4):

$$Y_t=DL_{1,t}{}^{1-\mu}\left[\int_0^K\left[\left[aL_{2,t}(i)\right]^{\alpha}\right]di\right]^{\frac{\mu}{\alpha}}$$

We followed Jones and Manuelli in the assumption that final product can be used either for consumption or as an input in R&D sector: $Y_t=C_t+R_t$, C_t being consumption, R_t being expenditures on R&D.

Number of intermediate goods evolves according to equation $K_{t+1}=K_t+BR_t$. One unit of final product enables to discover B units of new intermediate goods.

Jones and Manuelli use standard time-separable CRRA utility function

$$U=\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\chi}}{1-\chi}\right)$$

where β is a discount factor and χ is a coefficient of relative risk aversion.

Since total labour supply is fixed at $L=1$, market clearing condition on labour market is given by

$$L_{1,t}+\int_0^{K_t}L_{2,t}(i)\,di=1.$$

Jones-Manuelli model yields symmetric solution of $L_{1,t}=1-\mu$ and $L_{2,t}(i)=\mu/K_t$ for all i. Long term behaviour of the model depends on a relationship between μ and $\alpha/(1-\alpha)$.

Our model is a modification of a special case of Jones-Manuelli model where $\chi=1$ (i.e. utility is logarithmic) and $\mu=1$. We deviate from Jones and Manuelli in following manner:

Since our economy begins in period 0, it has no meaning to talk about allocation, say, $L_{4,t-3}$ for $t=1$. However, as we already mentioned, it is useful to think about initial endowments of producer goods of various orders as if produced in period –1. Then, by the same logic as when deriving (4), a term $\left(L_{2,t-1}{}^{\alpha}+\ldots+L_{K,t-K+1}{}^{\alpha}\right)^{\frac{1}{\alpha}}$ for $t=0$ is equal to initial endowment of producer goods of order 2, that is $Y_{2,-1}$. The sum $\left(L_{2,t-1}{}^{\alpha}+\ldots+L_{K,t-K+1}{}^{\alpha}\right)$ is equal to $\left(Y_{2,-1}\right)^{a}$. Analogically, $\left(L_{3,t-1}{}^{\alpha}+\ldots+L_{K,t-K+2}{}^{\alpha}\right)$ is equal to initial endowment of producer goods of order 3 to the power of *α*, that is $\left(Y_{3,-1}\right)^{a}$.

Total final output in period *t* is given by allocation of labour in the first-order sector in period *t*, in the second-order sector in period *t*–1 and so on.

As the total labour supply is equal to 1, market clearing condition on labour market takes the simple form:

$$L_t=\sum_{i=1}^{K}L_{i,t}=1 \tag{5}$$

Lagrangian of this problem is:

$$X=\sum_{t=0}^{\infty}\beta^t\ln\left(C_t\right)-\sum_{t=0}^{\infty}w_t\left(1-\sum_{j=1}^{K}L_{j,t}\right)=\sum_{t=0}^{\infty}\beta^t\ln\left(\varphi Y_{1,t}\right)-\sum_{t=0}^{\infty}w_t\left(1-\sum_{j=1}^{K}L_{j,t}\right)$$

φ – rate of consumption – constant in the short-run, w – Lagrange multiplier

However, as the ratio of the consumption to the total final output is constant in the short run, it is sufficient to solve[11]:

$$V=\sum_{t=0}^{\infty}\beta^t\ln\left(Y_{1,t}\right)-\sum_{t=0}^{\infty}w_t\left(1-L_{1,t}-L_{2,t}-\ldots-L_{K,t}\right) \tag{6}$$

Inserting short-run production function (4) into Lagrangian (6) yields:

$$V=\sum_{t=0}^{\infty}\beta^t\ln\left[\left(L_{1,t}{}^{\alpha}+L_{2,t-1}{}^{\alpha}+\ldots+L_{K,t-K+1}{}^{\alpha}\right)^{\frac{1}{\alpha}}\right]-\sum_{t=0}^{\infty}w_t\left(1-L_{1,t}-L_{2,t}-\ldots-L_{K,t}\right) \tag{7}$$

- Variable K_t (in Jones-Manuelli model understood as a number of intermediate goods, in our model interpreted as number of stages of production) evolves according to different function.
- Instead of a *continuum* of intermediate goods we use *set* of stages of production.
- Unlike Jones and Manuelli, we assume that labour allocated in different sectors (in Jones-Manuelli context in the production of different intermediate goods, in our model in different stages of production) enters production gradually. Instead of instantaneous production we assume time-consuming production.

The third modification leads to solution such that in the long run, economic growth will not occur irrespective of values of *α*. On the other hand, in Jones-Manuelli model, if $\alpha/(1-\alpha)$ is equal to μ (what in the case of $\mu=1$ amounts to the condition $\alpha=½$), economy endogenously grows by a constant rate. Furthermore, if $\alpha/(1-\alpha)>\mu$ (in the case of $\mu=1$ this condition reduces to $\alpha>½$) there is an explosive growth.

[11] This fact follows directly from the homotheticity of the utility function.

First order conditions with respect to labour can be expressed in the following way:

$$\beta^{t+i-1} \frac{1}{L_{1,t+i-1}{}^{\alpha} + L_{2,t+i-2}{}^{\alpha} + \ldots + L_{K,t+i-K}{}^{\alpha}} L_{i,t}{}^{\alpha-1} = w_t \qquad (8)$$

First order conditions are easy to interpret. Discounted marginal utility from marginal *final* product of labour in every sector has to be equal. In other words, discounted contribution of additional unit of labour in every sector to utility obtained from goods of order 1 has to be the same (as contribution to total final product of labour in *i*-th sector yields utility only after *i*-1 periods, it has to be *i*-1 times discounted.).

It follows from (8) that it is always efficient to use *all* stages of production that are available under current technology. Whatever the value of Lagrange multiplier w_t is, left-hand side of equation (8) is well-defined only if $L_{i,t} > 0$. In other words, as allocation of labour in any sector approaches zero, its marginal final product approaches infinity; therefore, it is optimal to employ labour in all sectors. If R&D is modelled as a separate sector, economic growth without progress in technology could not look like in Figure 3 – secular growth is not possible. On the other hand, with broader notion of structure of production, an investment in *R&D* is actually an investment in very high-order good. Economic growth can be thus imagined as in Figure 3, but growth of this kind does not satisfy Garrison's definition. We stress, that this conclusion does not hinge on the assumption $\lim_{L_i \to 0} dY_1 / dL_i = \infty$. If we relax this assumption and allow marginal final product of labour to approach finite positive value as allocation of labour in given sector approaches zero, it might be optimal not to allocate labour in early stages at all (irrespective of whether the economy is in short-run steady state or not). To see this, consider a general case of short run production function:

$$Y_t = \Phi\left(L_{1,t}, L_{2,t-1}, \ldots, L_{K,t-K+1}\right)$$
$$\Phi_{L_{i,t-i+1}}\left(\bullet\right) > 0; \Phi^2_{L_{i,t-i+1}}\left(\bullet\right) < 0$$

Assume that if no labour is allocated in sector *i*, marginal final product of labour in that sector does not approach infinity, but is finite and cannot be higher than some value δ_H. If all labour is allocated in sector *i* and no labour is allocated elsewhere, marginal final product of labour in *i*-th sector is minimal and it is equal to δ_L.

To maximize utility, following conditions would have to be satisfied:

$$\frac{d\Phi}{dL_{1,t}} = \beta \frac{d\Phi}{dL_{2,t-1}} = \ldots = \beta^{K-1} \frac{d\Phi}{dL_{K,t-K+1}}$$

It is straightforward to show that number of stages of production in use cannot exceed $\left(\ln\beta - \ln\delta_H + \ln\delta_L\right) / \ln\beta$. In other words, there is an upper limit on number of stages of production that it is optimal to employ. In the long run, secular growth as depicted in Figure 3 is not possible.

Now use the fact, that denominator of left-hand side of (8) is equal to $\left(Y_{1,t+i-1}\right)^{\alpha}$ to see that first order conditions (8) can be expressed as:

$$L_{1,t}=\frac{1}{m}\left(\frac{Y_{1,t+1}}{Y_{1,t}}\right)^{\frac{\alpha}{1-\alpha}}L_{2,t}=\frac{1}{m^2}\left(\frac{Y_{1,t+2}}{Y_{1,t}}\right)^{\frac{\alpha}{1-\alpha}}L_{3,t}=\ldots=\frac{1}{m^{K-1}}\left(\frac{Y_{1,t+K-1}}{Y_{1,t}}\right)^{\frac{\alpha}{1-\alpha}}L_{K,t} \tag{9}$$
$$m\equiv(1+\rho)^{\frac{1}{\alpha-1}}=(1+\rho)^{-\sigma}$$

m – rationing parameter

Rationing parameter *m* is always between zero and unity. From now on we will be interested only in the short-run steady state where $L_{i,t}=L_{i,t+1}$ and $Y_{i,t}=Y_{i,t+1}$ for all *i*'s[12] and *t*'s. Set of conditions (9) reduces to:

$$L_1=\frac{1}{m}L_2=\frac{1}{m^2}L_3=\ldots=\frac{1}{m^{K-1}}L_K \tag{10}$$

Combining with market clearing condition on labour market (5) yields the following expression for allocation of labour between different sectors:

$$L_1=\frac{1}{\sum_{i=1}^{K}m^{i-1}} \tag{11}$$
$$L_i=L_1m^{i-1}$$

Several important conclusions can be made from equations (10) and (11):

1. As elasticity of substitution approaches zero (*α* approaches –∞) and short-run production function approaches Leontief production function, *m* converges to 1. In this case labour is distributed uniformly between sectors irrespective of time preference ($L_1=L_2=\ldots=L_K=1/K$). Intuitively, if technology is Leontief, there is no freedom in allocation of labour between sectors. As we explain later, we do not consider this case to be relevant.
2. As elasticity of substitution approaches ∞ (*α* approaches 1), production function becomes linear. In this case first order conditions are no longer applicable and corner solution occurs with $L_1=1$ and $L_2=L_3=\ldots=L_K=0$. If there is perfect substitution between labour allocated in sector of order 1 and all other sectors, it is useless to allocate labour in sectors

[12] The transition to steady-state is fully governed by set of equations (8). Suppose that economy begins with no stock of producer goods of any order. Final product in the first period is thus less then in steady-state. As economy approaches steady state, final product grows. Therefore, the terms $\left(Y_{1,t+2}/Y_{1,t}\right)^{\frac{\alpha}{1-\alpha}}$ in (9) are always greater then unity. It follows that during the transition, relatively more labour is allocated in low-order sectors and relatively less labour is allocated in high-order sectors.

2 to K and *wait* until this labour yields utility. Instead, social planner should allocate all labour in sector of order 1 and utility is obtained instantaneously.

3. If technology is not Leontief and economic agents are not perfectly patient ($\rho > 0$), more workers are employed in late stages of production (sectors of lower orders) than in early stages of production (sectors of high orders).
4. If technology is not Leontief, fraction of labour allocated in sector of order 1 (and sectors of low orders in general) is increasing in time preference ρ. Our formal analysis thus confirms ATC (see Figure 2). With higher time preference, agents are less patient to wait for labour in high-order sectors to yield utility and therefore allocate labour in low-order sectors. Importantly, if agents are perfectly patient, i.e. $\rho = 0$, labour is distributed uniformly across sectors: $L_1 = L_2 = \dots = L_K = 1/K$. The logic behind is simple. In the absence of discounting, marginal product of labour in every sector is given by the same expression (8) (where $\beta = 1$). Therefore, to make marginal products equal, amount of labour allocated in every sector has to be the same. In this case, short-run production function reduces to:

$$Y_1 = K^{\frac{1-\alpha}{\alpha}} \tag{12}$$

Inserting (11) into production function (4) enables us to express total final output as a function of number of stages of production, coefficient of elasticity of substitution and coefficient of time preference:

$$Y_1 = \frac{1}{\sum_{i=1}^{K} m^{i-1}} \left[\sum_{i=1}^{K} m^{\alpha(i-1)} \right]^{\frac{1}{\alpha}}$$

After the summation of geometric series $\sum_{i=1}^{K} m^{\alpha(i-1)}$ and $\sum_{i=1}^{K} m^{\alpha(i-1)}$ we obtain:

$$Y_1 = g(K) = \frac{1-m}{1-m^K} \left(\frac{1-m^{\alpha K}}{1-m^{\alpha}} \right)^{\frac{1}{\alpha}} \tag{13}$$

We will refer to this function as *long-run production function.* Long-run production function is schematically depicted in Figure 4, its first derivative – marginal final product of additional stage of production – is depicted in Figure 5. In the rest of this section, we focus on the different characteristics of long-run production function that will be of great importance in analysis of long-run behaviour of the model.

First observe that marginal final product of additional production stage is given by:

$$\frac{dY_1}{dK} = g'(K) = \left[\frac{(1-m)\ln m}{(1-m^{\alpha})^{\frac{1}{\alpha}}}\right]\left[\frac{(1-m^{\alpha K})^{\frac{1}{\alpha}}}{1-m^K}\right]\left[\frac{m^K}{1-m^K} - \frac{m^{\alpha K}}{1-m^{\alpha K}}\right] \tag{14}$$

As the first term is always negative and the second term is always positive, sign of the marginal product depends on the sign of the third term. If $0 < \alpha < 1$, value of $m^K / (1-m^K)$ is lower than value of $m^{\alpha K} / (1-m^{\alpha K})$ (both being positive) and marginal product is positive. If $\alpha < 0$, value of $m^{\alpha K} / (m^{\alpha K} - 1)$ is positive while $m^K / (m^K - 1)$ is negative, marginal product is therefore negative.

Figure 4
Long-run production function

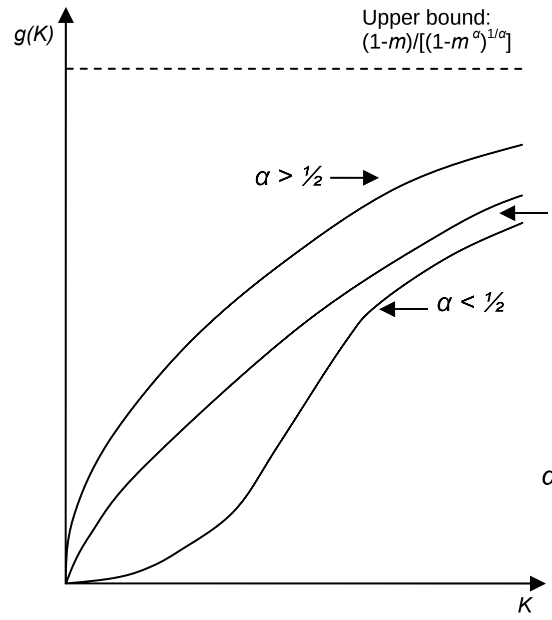
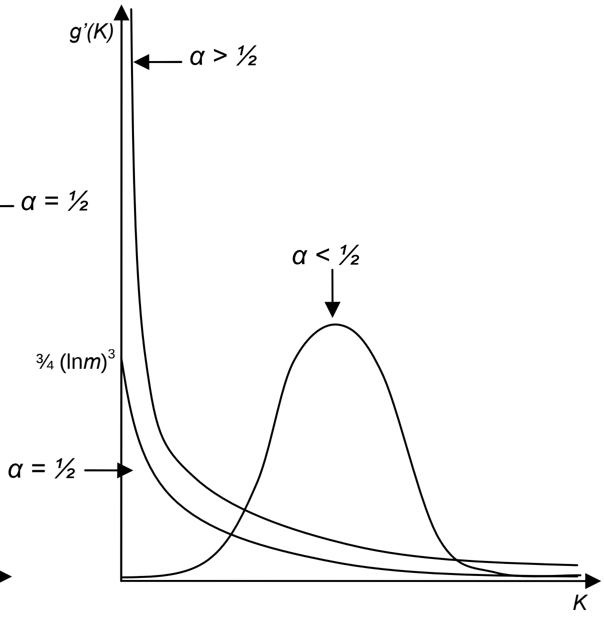


Figure 5
Marginal final product



If elasticity of substitution σ is higher than unity ($\alpha > 0$), additional production stage enables more efficient allocation of labour and more total product. On the other hand, if σ is less than one ($\alpha < 0$) higher number of stages of production leads to less product. This can be easily understood in the case of elasticity of substitution approaching zero (α approaching $-\infty$). Short-run production function becomes Leontief, i.e. $Y_1 = \min(L_1, L_2, ..., L_K)$. To maximize product (maximum product in this case coincides with maximum utility), labour is distributed uniformly across sectors: $L_1 = L_2 = ... = L_K = 1/K = Y$. More sectors means *more complicated* but not more efficient technology and Austrian economic growth based on lengthening of production chain is not possible.

To analyze an extreme Cobb-Douglas case $\alpha = 0$, observe that in the steady state, if $\alpha = 0$, production function (4) can be expressed as:

$$Y_1 = L_1^{\frac{1}{K}} L_2^{\frac{1}{K}} \dots L_K^{\frac{1}{K}} = L_1 \left(m^{\frac{1}{K}} m^{\frac{2}{K}} \dots m^{\frac{K-1}{K}} \right) = \frac{1}{\sum_{i=1}^{K} m^{i-1}} m^{\frac{K-1}{2}}$$

Once we sum up $\sum_{i=1}^{K} m^{i-1}$, Cobb-Douglas production yields long-run production function of the following form:

$$Y_1 = \frac{1-m}{1-m^K} m^{\frac{K-1}{2}} \tag{15}$$

Production function (15) is similar to (13) ($\left[\left(1-m^{\alpha K}\right)/\left(1-m^{\alpha}\right)\right]^{\frac{1}{\alpha}}$ in (13) is replaced by $m^{\frac{K-1}{2}}$ in (15)). First derivative of (15) with respect to *K* is also similar to (14) (once again $\left[\left(1-m^{\alpha K}\right)/\left(1-m^{\alpha}\right)\right]^{\frac{1}{\alpha}}$ is replaced by $m^{\frac{K-1}{2}}$):

$$\frac{dY_1}{dK} = \left[(1-m)\ln m\right] \left[\frac{m^{\frac{K-1}{2}}}{1-m^K} \right] \left[\frac{m^K}{1-m^K} + \frac{1}{2} \right]$$

It is easy to see that the value of the first derivative is always negative (the first term is always negative, the second and the third term are always positive).[13]

If Austrian stages of production are to be formalized as a sequence of CES-production functions, assumption $\alpha > 0$ which implies elasticity of substitution greater than unity is necessary. Therefore, from now on we focus only on the case $\alpha > 0$.

Character of the second derivative of the total product with respect to the number of stages of production also depends on the parameter *α*.

- If elasticity of substitution is greater or equal to 2 ($\alpha \geq ½$) second derivative is always negative. For *K* close to zero, marginal final product of additional stage of production

[13] Numerical example illustrates why with Cobb-Douglas technology final product falls with increasing number of stages of production. Assume that there is only one stage of production. Production function reduces to $Y_1 = L_1$. Allocation of labour in sector of order 1 is $L_1 = 1$; product is equal to $Y_1 = L_1 = 1$. Now increase the number of stages of production to 2. Production function is $Y_1 = L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}$, under the constraint $L_1 + L_2 = 1$ the highest attainable product is $Y_1 = \left(\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{2}$. Analogically if the number of stages of production is equal to 3, the highest attainable product is $Y_1 = \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{3}\right)^{\frac{1}{3}} = \frac{1}{3}$.
Actually, in the case of perfect patience $\rho = 0$ where labour is distributed uniformly across sectors $L_1 = L_2 = \dots = L_K = 1/K = Y$, Cobb-Douglas production reduces to strictly decreasing function $Y_1 = 1/K$.

is very high, $\lim_{K\to 0} g'(K) = \infty$. As *K* grows, $g'(K)$ falls and gradually approaches zero, $\lim_{K\to 0} g'(K) = 0$.

- If σ is less than 2 ($\alpha < ½$) second derivative is positive for low values of *K,* however, with growing *K* it becomes negative. For low values of *K*, marginal final product of additional stage of production is very low, $\lim_{K\to 0} g'(K) = 0$. With increase in *K* we at first observe increasing marginal final product, after some point marginal final product begin to decline and gradually approach zero, $\lim_{K\to 0} g'(K) = 0$.
- If σ is equal to 2 ($\alpha = ½$) second derivative is always negative, however, as *K* approaches zero, $g'(K)$ does not approach infinity (as in the case $\alpha \geq ½$), but positive constant, $\lim_{K\to 0} g'(K) = (3/4)(\ln m)^3$. As *K* grows to infinity, $g'(K)$ approaches zero, $\lim_{K\to 0} g'(K) = 0$.

Underlying mechanics is easy to understand in extreme case of perfect patience, where $\rho = 0$ and $m = 1$. As already mentioned, in this case, short-run production function is reduces to $Y_1 = K^{\frac{1-\alpha}{\alpha}}$. For $\alpha > ½$, exponent of $K^{\frac{1-\alpha}{\alpha}}$ is lower than unity, which means that diminishing marginal returns to number of stages of production applies. If $\alpha < ½$, increasing marginal returns are observed – the second derivative is positive. If elasticity of substitution is equal to 2 ($\alpha = ½$), production exhibits constant marginal returns. However, if $\rho > 0$ and $m < 1$ labour is not distributed uniformly across sectors. The higher the stage of production, the less labour is allocated in the sector because of the growing delay between exerting labour and obtaining utility. Deviating more and more from $L_1 = L_2 = ... = L_K = 1/K$ creates tendency for diminishing marginal returns. In the case of $\alpha > ½$, already existing decreasing of marginal product is emphasized. If $\alpha < ½$, after some critical number of stages of production, increasing marginal returns are completely offset by deviating from uniform distribution of labour and diminishing marginal product applies. If $\alpha = ½$, constant marginal returns never have chance to be observed, marginal product is always decreasing.

Finally, there is one more peculiarity about the long-run production function (12). As the number of stages of production grows to infinity, product *does not* grow without boundary.

$$\lim_{K\to\infty} Y_1 = \lim_{K\to\infty} \frac{1-m}{1-m^K}\left(\frac{1-m^{\alpha K}}{1-m^{\alpha}}\right)^{\frac{1}{\alpha}} = \frac{1-m}{\left(1-m^{\alpha}\right)^{\frac{1}{\alpha}}} \tag{16}$$

When the number of stages of production is high, very little labour is allocated in the highest stages of production because agents are not willing to wait so much for the labour there allocated to yield utility. Inventing new stage of production has thus almost no impact on total product because very little labour is allocated in an additional stage. Impatience thus constitutes upper boundary on total product. As agents become perfectly patient (time preference ρ approaches 0, rationing parameter *m* approaches 1) upper boundary cease to exist.

$$\lim_{K\to\infty, m\to 1^-} Y_1 = \lim_{m\to 1^-} \frac{1-m}{\left(1-m^\alpha\right)^{\frac{1}{\alpha}}} = \infty$$

3.2. Finding a long-run equilibrium

Expression (16) has decisive impact on the long-run implications of the model. Because of the impatience of economic agents, growing number of stages of production cannot be the source of endogenous economic growth. Even if the number of stages of production grew exogenously, long-run growth would have not been possible.

Furthermore, it will be shown that economic growth will eventually stop after reaching certain number of stages of production, or in other words, there is a long-run steady state where number of stages of production *K* is kept constant.

To simplify our analysis, we assume that *K* can take also *non-integer* values. From the long-run point of view, economy is always in short-run steady state and total product is given by the long-run production function $g(K)$ (13). Total output is divided between consumption and expenditures on R&D. In the long run, next-period number of stages of production depends on expenditures on R&D. Since we have already relaxed our assumption that *K* can take only *integer* values, we do not assume that next-period *K* will be equal to *K*+1 with probability *P* given by (2) and to *K* with probability 1–*P* , instead, we assume that next-period *K* is equal to $K+P$. This simplification is made mainly for analytical reasons. Furthermore, in the long-run, we set up the model in the continuous time. Dynamics of *K* is thus given by

$$\dot{K} = B\left\{1-e^{-A_L\left[g\left[K(t)\right]-C(t)\right]}\right\} \tag{17}$$

A_L – long-run efficiency coefficient

where $R(t) = g\left[K(t)\right] - C(t)$. Since expenditures on R&D cannot be negative (in other words, consumption cannot exceed total product), inequality $C(t) \le g\left[K(t)\right]$ must hold.

Several simplifying assumptions allow us to formulate simple optimal control problem. Combining function (17) with utility function (3) yields following present-value Hamiltonian:

$$J = e^{-\rho_L t}\ln\left[C(t)\right] + \lambda(t)B\left\{1-e^{-A_L\left[g(K(t))-C(t)\right]}\right\} \tag{18}$$

with constraint

$$C(t) \le g[K(t)]$$

a transversality condition:

$$\lim_{t\to\infty} \lambda(t)K(t) = 0 \tag{19}$$

ρ_L – long-run time preference; λ – costate variable

It is important to note that one short-run period (say, 1 year) is not equal to one period in the long-run, as in the long-run economy is always in short-run steady state. If, for instance, it takes 10 years for economy to approach close enough to short-run steady state, one long-run period has to be identified as at least 10 years. Therefore, it is not possible to use same value ρ as time preference in the short and in the long run. We use ρ_L to denote long-run time preference, ρ_L being higher than ρ. The same applies for efficiency coefficient A in R&D production function. In the long run, we use A_L instead of A, A_L being higher than A. We emphasise in advance that the characteristics of a steady state depend on the fraction ρ_L/A_L. Because of that, multiplication of ρ and A by the same constant has no effect on the steady state. Furthermore, assumption of one long-run period being equal to, say, 10 short-run periods implies that in one long-run period agents consume ten times more than in one short-run period. However, it is clear that Hamiltonian

$$J' = e^{-\rho_L t}\ln\left[10\times C(t)\right] + \lambda(t)B\left\{1 - e^{-A_L\left[g(K(t))-C(t)\right]}\right\}$$

yields the same solutions as Hamiltonian (18).

To solve the optimal control problem stated by (18) and (19) proceed as follows: calculate the first order condition with respect to control variable C

$$\lambda = \frac{1}{A_L BC} e^{-\rho_L t + A_L\left[g(K)-C\right]} \tag{20}$$

and the first order condition with respect to state variable K:

$$\frac{\dot{\lambda}}{\lambda} = -g'(K)A_L Be^{-A_L\left[g(K)-C\right]} \tag{21}$$

Now take the derivative of (20) with respect to time and divide by (20) to yield an expression for growth rate of costate variable *λ:*

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{C}}{C} - \rho_L + A_L g'(K)\dot{K} - A_L\dot{C} \tag{22}$$

Use the equality of (21) and (22) and substitute for dK/dt from (17). After some manipulation, expression for dynamics of C can be obtained:

$$\dot{C} = \left[A_L Bg'(K) - \rho_L\right]\frac{C}{1 + A_L C} \tag{23}$$

This expression, together with (17) and transversality condition (19) describes dynamics of the system. Values of K and C in steady state where $dK/dt = 0$ and $dC/dt = 0$ can be easily calculated by equating (18) and (23) to zero:

$$C^* = g\left(K^*\right) \tag{24}$$

For consumption to be constant following equation must hold:

$$A_L B g'\left(K^*\right) - \rho_L = 0.$$

As $A_L/\rho_L = A/\rho$, we can write:

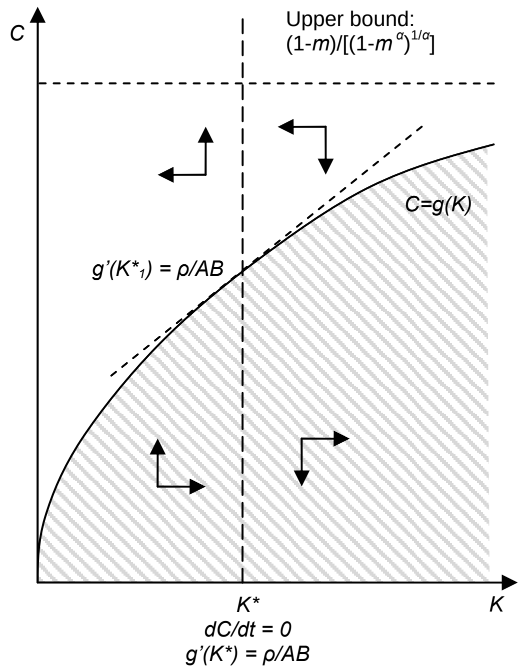
$$ABg'\left(K^*\right) - \rho = 0. \tag{25}$$

Alternatively:

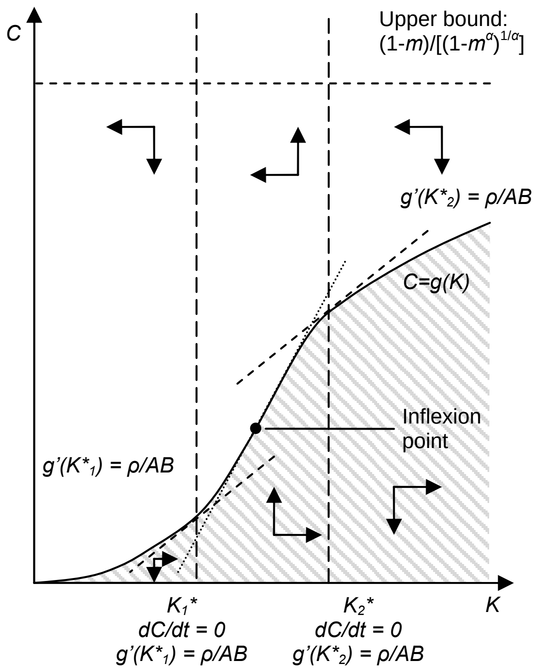
$$K^* = g'^{-1}\left(\frac{\rho}{AB}\right) \tag{26}$$

where g'^{-1} is the inverse of the first derivative of the long-run production function (13). However, g'^{-1} exists only if g' is strictly decreasing, that is $\alpha \geq ½$. Phase diagram for $\alpha > ½$ is depicted in Figure 6. There is one steady state.

F i g u r e 6
One steady state; α > ½



F i g u r e 7
Two steady states; a < 1/2



Shaded area – set of feasible controls

Let us now examine three different possible cases.

Case A: α > ½

This is the simplest case. Since $\lim_{K\to 0} g'(K)=\infty$, $\lim_{K\to\infty} g'(K)=0$ and $g'(K)$ is continuous and strictly decreasing, there is always exactly one steady-state. Phase diagram suggests that system is saddle-path stable. [14] If initial value of K is lower than K^*, economic agents choose such a value of consumption that the economy finds itself on stable arm and gradually approaches steady state. If initial value of $K(0)$ is higher than K^*, economic agents would like to choose consumption higher then total product g(K) and 'eat' some K, however, as K represents technology, this is not possible. In other words, constraint $C(t)\leq g\left[K(t)\right]$ is binding (in Figure 6 and 7, set of feasible controls is represented by shaded area) and corner solution occurs with $C=g(K)$.

Case B: α < 1/2

In this case, because of the convex-concave character of function $g(K)$, if the product of two coefficients of efficiency of R&D sector AB is sufficiently high, equation (25) holds for two values of K and there are two steady states – see Figure 7. Steady state where $g''(K^*)<0$ is saddle-path stable (higher value of K^*), on the other hand, steady state where $g''(K^*)>0$ is unstable (lower value of K^*).[15]

However, as AB declines, value of the ratio ρ/AB increases and two steady states approach each other. If ρ/AB is equal to the value of the first derivative of g at the point where g is the steepest, i.e. at inflexion point, there is only one steady state. From the left-hand side, system is unstable, from the right-hand side, system is saddle-path stable.[16]

If AB is even lower, there is no inner solution for the long-run problem and corner solution occurs, where $C=g(K)$ in every period.

---

[14] This can be verified by examination of Jacobian matrix **Z** made of partial derivatives of (18) and (23) with respect to C and K at steady states.

$$\mathbf{Z}=\begin{pmatrix} 0 & \dfrac{C^* A_L B g''(K^*)}{1+A_L C^*} \\ -A_L & \dfrac{\rho_L}{B} \end{pmatrix}$$

In the case of $g''(K^*)<0$, both eigenvalues of **Z** are real, one being positive, one being negative – system is saddle-path stable.

[15] If $g''(K^*)>0$, both eigenvalues of **Z** are either real and positive or complex with positive real parts – system is unstable. If $g''(K^*)<0$, both eigenvalues of **Z** are real, one being positive, one being negative – system is saddle-path stable.

[16] In the case of If $g''(K^*)=0$, one eigenvalue is real and positive, another is equal to zero. Steady-state is knot-saddle point.

Case C: α = 1/2

Case of $\alpha = ½$ is similar to the case $\alpha > ½$. A difference is in the fact, that value of the first derivative of function g at K=0 is finite and is equal to $\lim_{K\to 0} g'(K) = (3/4)(\ln m)^3$. Because of that, if ρ/AB is higher then $\lim_{K\to 0} g'(K)$ there is again no solution to the long-run problem.

Analysis of long-run problem suggests that if the efficiency of R&D sector is too low, there might be no equilibrium because marginal product of additional stage of production is never high enough to encourage costly investment in R&D. Because of this, it is desirable to calculate minimal level of R&D efficiency that ensures the existence of equilibrium. This amounts to find an inflexion point of the long-run production function g for various α in interval $\left(0, ½\right\rangle$ and ρ (ρ influences value of m) and to calculate the slope of a tangent to g at this points. In other words, we need to find maximal value of the first derivative of the function g for various α and ρ. Denote this value g'_{MAX}. Minimal value of product AB which ensures the existence of at least one equilibrium is given by ρ/g'_{MAX}. If AB is higher than ρ/g'_{MAX}, there are two equilibriums. Even if it is not possible to solve this problem analytically, maximal value of g' can be found numerically. We report critical values of AB for various α and ρ in

Table 1
Minimal values of AB ensuring existence of equilibrium for α ≤ ½

	ρ_1 = 0,02	ρ_2 = 0,04	ρ_3 = 0,06	ρ_4 = 0,08	ρ_5 = 0,1
α_1 = 0,05	5,58E-57	2,45E-51	4,58E-48	9,13E-46	5,36E-44
α_2 = 0,15	5,96E-14	1,26E-12	1,19E-11	5,84E-11	1,98E-10
α_3 = 0,25	5,2E-07	4,08E-06	1,35E-05	3,14E-05	6,03E-05
α_4 = 0,35	0,000335	0,001202	0,002531	0,004282	0,006428
α_5 = 0,45	0,008146	0,018963	0,031055	0,049348	0,057715
α_6 = 0,49	0,017569	0,036129	0,055068	0,074247	0,093598
α_7 = 0,50	0,019999	0,039995	0,059983	0,079961	0,099924

We observe that requirements on R&D efficiency increase with α. The lower α, the faster marginal product of additional stage of production increases. With lower α, at the moment when tendency for decreasing marginal product prevails, i.e. at inflexion point, marginal product is higher. As marginal product is higher, lower efficiency of R&D is needed to ensure profitable investment in R&D. Furthermore, requirements on R&D efficiency increase with ρ.

If economic agents are less patient, higher efficiency of R&D is needed to make them postpone current consumption and invest in R&D.

In this respect our analysis yields results very different from the results of Jones and Manuelli. Even if the technology allows for increasing returns to scale ($\alpha < ½$), not only there is no explosive growth, but there might be no inner solution at all. If efficiency of R&D is too low, it might be always optimal to consume entire final product.

At this point we focus on economic interpretation of steady-state conditions. Assume that R&D efficiency is high enough to ensure existence of inner solution. The condition (23) is straightforward: number of stages of production is constant only if no resources are invested in R&D, i.e. whole output is consumed. To interpret the condition for constant consumption (25), observe first that (24) implies $A_L Bg'(K^*)/\rho_L = 1$. In equilibrium, economic agents cannot by definition increase their utility by consuming slightly less today, investing amount saved in R&D and then consume more in all future periods when the number of stages of production is higher (alternatively, it is not possible to increase utility by consuming slightly more). Now imagine that agents decrease their consumption by 1 unit. Current utility therefore decreases by $u'(C^*)$. By investing one more unit of total product in R&D, next-period number of stages of production increase by a value given by derivative of (17) with respect to *R,* that is $A_L Be^{-A_L R^*}$ or $A_L Be^{-A_L[g(K^*)-C^*]}$. This expression gives marginal product of R&D expenditures in terms of the number of stages of production. As one more stage of production enables to increase total product by $g'(K^*)$, increase in the number of stages of production by $A_L Be^{-A_L[g(K^*)-C^*]}$ enables to increase total product by $g'(K^*)A_L Be^{-A_L[g(K^*)-C^*]}$. Total utility in all following periods thus increases by $u'(C^*)g'(K^*)A_L Be^{-A_L[g(K^*)-C^*]}$. In equilibrium, 'current value' of additional utility $u'(C^*)g'(K^*)A_L Be^{-A_L[g(K^*)-C^*]}$ has to be equal to forgone utility given by $u'(C^*)$ which yields the condition:

$$u'(C^*) = \frac{u'(C^*)g'(K^*)A_L Be^{-A_L[g(K^*)-C^*]}}{\rho_L}$$

Now use the fact that in steady state all output is consumed to see:

$$\frac{A_L Bg'(K^*)}{\rho_L} = 1$$

In addition to the fact that impatient agents are willing to allocate only very little labour in very high-order sectors which makes additional sectors almost unproductive, there is a second upper limit on final product. There is a value of the number of stages of production when additional stage of production enables to increase final product only by so little that discounted additional utility which is thus obtained does not compensate for the costs of inventing it. The discount factor is too high, agents are too impatient.

Observe that the number of stages of production and the level of output grows as rate of time preference decreases.[17]

Level of output is also increasing in efficiency of R&D expressed by parameters *A* and *B*. With higher efficiency of R&D, it is less costly to invent new stage of production; it is therefore optimal to keep investing in R&D for a longer time.

4. POTENTIAL SOURCES OF ENDOGENOUS GROWTH

We have shown that under plausible assumptions, economic growth based solely on accumulation of producer goods and growing number of stages of production might not be possible. In other words, once time-consuming production is introduced into standard model of technological change, endogenous growth cease to exist. In this section we propose several modifications of the model that will make endogenous growth possible. We briefly mention human capital and labour-augmenting technological progress and we investigate possible effects of production-accelerating technological progress.

4.1. Human capital

As firstly shown by Lucas (1988) and by many others later on, human capital might be one of the most important sources of the economic growth. It is possible to incorporate human capital in our model in the same way as Lucas incorporates human capital in neoclassical model of economic growth.

We will modify set of production functions (1) in the following way: assume, that volume of capital goods of *i*-th order $Y_{i,t}$ is function of volume of capital goods of order *i*+1 (i.e. $Y_{i+1,t-1}$) and *human capital* $H_{i,t}$ employed in *i*-th sector:

$$Y_{i,t} = \left(Y_{i+1,t-1}{}^{\alpha} + H_{i,t}{}^{\alpha}\right)^{\frac{1}{\alpha}} \text{ for } i < K$$
$$Y_{i,t} = H_{i,t} \text{ for } i = K$$

Short run production function (4) changes into:

$$Y_{1,t} = \left(H_{1,t}{}^{\alpha} + H_{2,t-1}{}^{\alpha} + \ldots + H_{K_t,t-K+1}{}^{\alpha}\right)^{\frac{1}{\alpha}} \tag{27}$$

In the short run, stock of human capital is constant and is equal to *H*. Economic agents devote fraction *v* of human capital to production of consumer (and producer) goods and remaining $(1-v)$ to further accumulation of human capital (actually, it would be more promising to assume that part of human capital is allocated in R&D sector as well; for simplicity, we

[17] However, contrary to ATC, time preference of economic agents does not influence long-run rate of growth which is equal to zero. In other words, similarly to neoclassical model, changes in time preference have level effects only and no growth effects.

abstract from this possibility at this point). Define $h_{i,t}$ as ratio of human capital employed in sector *i* (i.e. $H_{i,t}$) to the total stock of human capital employed in production sector vH, that is $h_{i,t} = H_{i,t}/vH$. Production function (27) can be written as:

$$Y_{1,t} = vH\left(h_{1,t}{}^{\alpha} + h_{2,t-1}{}^{\alpha} + \ldots + h_{K,t-K+1}{}^{\alpha}\right)^{\frac{1}{\alpha}}$$

Optimal values of $h_{i,t}$ will be chosen by the same way as values of $L_{i,t}$. Long-run production function (13) now takes form:

$$Y_1 = vH\frac{1-m}{1-m^K}\left(\frac{1-m^{\alpha K}}{1-m^{\alpha}}\right)^{\frac{1}{\alpha}} \tag{28}$$

We may assume that in the long run, accumulation of human capital is governed by familiar equation

$$\dot{H}(t) = H(t)D\left[1-v(t)\right]$$

where *D* is maximum rate of human-capital accumulation.

Since the long-run production function (28) exhibits constant returns to produced input *H* (human capital) it follows that long-run endogenous growth with solution similar to those of Lucas (1988) is possible. In Figures 6 and 7, growing stock of human capital will be accompanied by steepening of the loci $C = g(K)$. Steady states will therefore 'shift' to the right and the number of stages of production will rise with growing stock of human capital.

Observe that it would be a mistake to consider increasing number of stages of production to be a source of long-run economic growth. The only true source of long-run growth is growing stock of human capital. This is what makes investment in increasing number of stages of production effective. [18] Implications of this on Austrian views on economic growth are extremely important. It might be the case, that lengthening of horizontal leg and vertical leg of Hayekian triangle (see Figure 3) are to the great extent independent from each other and are caused by the third factor – for instance by growing stock human capital. Of course, as in the case of R&D, we can include investment in human capital *inside* the Hayekian triangle. Then, investment in human capital would have to be understood as investment in producer goods of very high order and to invest in human capital would actually mean to expand horizontal leg of the triangle. The term *'capital'* in *'Austrian theory of capital'* would have to be understood in its broader sense including human capital.

[18] We would make precisely the same mistake if we considered investment in physical capital to be the source of long-run growth in the neoclassical model of economic growth. The only true source of the growth in neoclassical model is exogenous technological progress.

4.2. Labour-augmenting technological progress

Alternatively, assume that there is labour-augmenting technological progress, the level of technology being constant in the short run. The production function (4) is modified in the following manner:

$$Y_{i,t} = \left[Y_{i+1,t-1}{}^{\alpha} + \left(TL_{i,t}{}^{\alpha} \right) \right]^{\frac{1}{\alpha}} \text{ for } i < K$$
$$Y_{i,t} = TL_{i,t} \text{ for } i = K$$

T – coefficient of technology

Modification of long-run production function is straightforward:

$$Y_1 = T \frac{1-m}{1-m^K} \left(\frac{1-m^{\alpha K}}{1-m^{\alpha}} \right)^{\frac{1}{\alpha}} \tag{29}$$

Due to constant returns to the level of technology *T*, an endogenous growth will be possible. Detailed behaviour of an economy would depend on the way technological progress happens.

4.3. Production-accelerating technological progress

Human capital and labour-augmenting technological progress are sources of endogenous growth somewhat unrelated to ATC. However, it seems ATC enables us to identify one specific potential source of endogenous growth.

Imagine that technological progress allows 1 worker to produce 1 unit of output twice as fast. We will refer to this type of technological progress as *production-accelerating*. In neoclassical concept where production is instantaneous it means that in 1 unit of time 1 worker produces twice as much output. Production-accelerating technological progress amounts to labour-augmenting technological progress.

In ACT context, there is a clear distinction between two types of technological progress. Effects of labour-augmenting technological progress have been already described. On the other hand, ability to produce 1 unit of output twice as fast means that turning producer goods of order *i* into goods of order *i*-1 takes only ½ period instead of 1.

To model such technological changes is somewhat awkward. However, little trick proves to be useful. Observe that if households need to wait for something, to make world run twice as fast amounts to become twice as patient. It is therefore possible to investigate properties of short-run steady state by using $\rho/2$ instead of ρ. If initially we identified one period with, say, one year, now one period amounts to half a year. If Y_1 (which denotes output *per one period*) rises after switching from ρ to $\rho/2$, it means that in a half year 'fast' economy is able to produce more than ‘slow’ economy in a whole year. To see that output per one period in fast economy is really higher than in slow one, define *s* as speed of production. Characteristics of steady-states under various speeds of production can be examined by using ρ/s instead of ρ. With such a modification, rationing parameter *m* is given by:

$$m \equiv \left(1+\frac{\rho}{s}\right)^{-\sigma} \tag{30}$$

Since *m* increases in *s,* as speed of production grows to infinity, *m* approaches unity. Observe that for $K > 1$, the derivative of the long run production function (13) with respect to *m* is positive. Intuitively, if speed of production increases, it is not necessary to wait so long until labour in early stages of production yields utility. Therefore, it is optimal to allocate more labour in those stages than before and distribution of labour between various sectors approaches uniform distribution which increases total final product. This ‘levelling’ effect, coupled with the fact that what was one year is now half a year causes that doubling the speed of production has more than proportionate effect on production. To see both effect of an increase in the speed of production, insert (30) in (13) to express the former and multiply by *s* to express the latter. We obtain long-run production function:

$$Y_1 = s\frac{1-\left(1+\frac{\rho}{s}\right)^{-\sigma}}{1-\left(1+\frac{\rho}{s}\right)^{-\sigma K}}\left(\frac{1-\left(1+\frac{\rho}{s}\right)^{-\sigma\alpha K}}{1-\left(1+\frac{\rho}{s}\right)^{-\sigma\alpha}}\right)^{\frac{1}{\alpha}}$$

Compare the changes in the speed of production to labour-augmenting technological progress. While the acceleration of production has more than proportionate effect on production, it follows from (29) that an increase in coefficient of technology has only proportionate effect on production. Changes in *s* have greater effect on the production function $g(K)$ than changes in *T.*

To see the long run effects of an increase in the speed of production, observe that in Figure 6 and 7, growing speed of production would cause loci $C = g(K)$ to be steeper. Steady states (for given values of speed of production *s*) ‘shift’ to the right and the economy grows Furthermore, increased speed of production means that labour in early stages is more productive, it is therefore optimal to invest in R&D and attempt to invent new stages of production. Thus, introduction of possibility of increasing speed of production (for instance via second R&D sector) would therefore allow for endogenous growth in the model. It is ATC perspective that enables us to identify this source of economic growth.

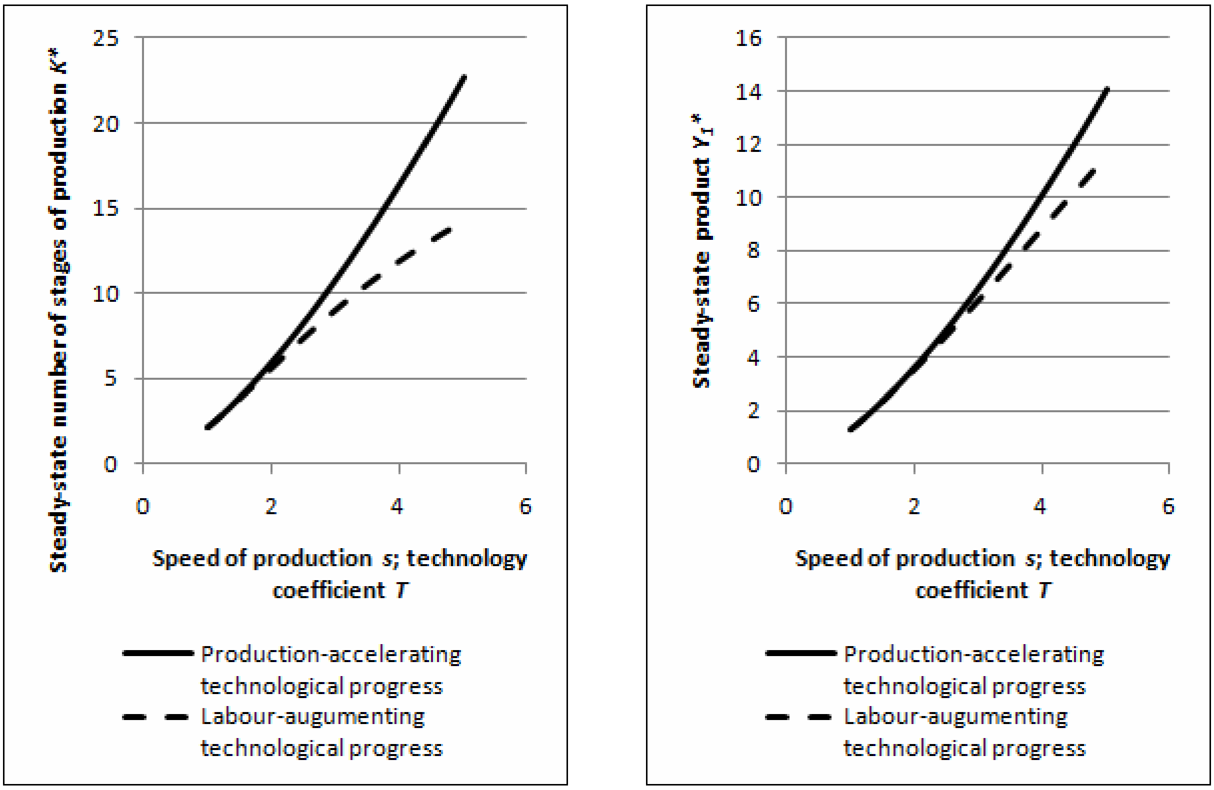
Recall our example with the abacus and the calculator. Since it would take years or decades to construct the calculator in medieval ages, instructions how to do that would be almost useless. If scholars were offered to *buy* such a textbook, they would not be willing to pay much for it. In modern terms, it would not be efficient to invest in such a technology. On the other hand, in early 1900, the instructions to construct modern calculator would be extremely useful as it would not take so long to do that. It would be profitable to pay even a very high price for the textbook. Even if investment in new technology may not pay off in a slow economy, it may pay off in a fast economy.

Moreover, as an increase in s shifts production function $g(K)$ more dramatically than an increase in T, production-accelerating technological progress has greater long-run effects than labour-augmenting technological progress.

Figures 8 and 9 provide numerical examples of different effects of two types of technological progress. Value of speed of production s and value of technology coefficient T appear on the horizontal axis. On vertical axis, steady-state number of stages of production K^* (Figure 8) and steady-state product Y_1^* (Figure 9) appear. In the case of production-accelerating technological progress, we let s move from 1 to 5, in the case of labour-augmenting technological progress we let T move in the same interval. Numerical examples illustrate greater effect of an increase in the speed of production on steady-state value of number of stages of production and on steady-state product as opposed to an increase in the technology coefficient.

F i g u r e s 8 and 9

Two types of technological progress ($\alpha = 0.75$; $\rho = 0.04$; $A = 1$; $B = 0.2$; $\rho_L/A_L = \rho/A = 0.04$)



Of course, in reality, one cannot distinguish between different types of technological progress. Much more production stages might be needed to produce durable capital goods which allow acceleration of production. Think about assembly line, railway, telegraph or already mentioned calculator. On the other hand, production-accelerating technological progress might be disembodied at might consist in innovative managerial techniques (e.g. just-in-time).

As for the practical conclusions, our paper suggests following hypothesis: Apart from increasing the productivity of labour, industrial and managerial revolution might have had different influence on further economic development. Accelerating of production processes they allowed was a necessary condition for further investment in R&D to be effective. In this way, one investment in R&D fostered another.

CONCLUSION

The purpose of our paper is to analyse the issue of economic growth based on technological change if production is time-consuming. Inspired by Austrian theory of capital, we modified model of Jones and Manuelli (1997), section 6. We formalized Austrian stages of production as a sequence of CES-production functions and solved for short-run equilibriums for given number of stages of production as well as for long-run equilibriums when the number of stages of production grows as a result of R&D. Our analysis showed that because of impatience of economic agents, relatively less labour is allocated in high stages of production in comparison to low stages of production. Because of this, as the number of stages of production grows to infinity, total product converges to finite value. Further more, once the number of stages of production reaches certain value, marginal utility of additional stage of production is too low to motivate further investment in R&D, therefore, the economy converges to steady-state and endogenous growth is not possible. Moreover, if marginal final product of labour approaches infinity as allocation of labour in given stage approaches zero, it is always optimal to use *all* stages of production that are available under current technology (if it approaches finite value, there is an upper limit on the number of stages of production that it is optimal to use).

We propose several remedies to this ‘problem’, traditional ones being accumulation of human capital and endogenous labour-augmenting technological progress. However, Austrian viewpoint motivates third solution – production-accelerating technological progress. If it is possible to increase a speed of production, i.e. to shorten the period needed to transform goods of order i to goods of order i-1, endogenous growth is possible.

Our analysis hinges on the formalization of stages of production as a sequence of production functions with constant elasticity of substitution higher than unity. However, it is not clear whether it is useful to think about labour and producer goods of order i as being substitutable with elasticity higher than 1 in the production of goods of order i-1. Intuition may even tell in favour of Leontief technology. Therefore, it is one of the major challenges to Austrian economists to provide details of the technology on which Austrian theory of capital is founded – namely the technical details of how lengthening of production process can be reflected in the increase in final product.

In spite of the fact that our analysis provides an argument against the possibility of secular growth as defined by Roger W. Garrison, incorporation of Austrian theory of capital into the theory of economic growth may lead to opening up of new areas of research. Relationships between accumulation of human capital, various kinds of technological progress with emphasis on increasing speed of production and structure of physical capital are ones of them. Practical necessity of the study of these phenomena is hardly to be overemphasized. In the light of Austrian theory of economic cycle as elaborated by Friedrich A. von Hayek, deeper understanding of capital structure and structure of production is one of the key elements of our understanding not only of economic growth, but of business cycles in growing economies as well.

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