Income-oriented retirement investment model

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Abstract

In this paper, we present a stochastic dynamic model of retirement savings with the maximization of expected utility of retirement income. In contrast to traditional models, we change the paradigm from value-oriented to income-oriented. We use SeLFIES (Standard-of-Living Forward-starting Income-only Securities) as risk-free assets. We show that our approach leads to a change in the risk ranking of assets and less risky saving strategies.

Introduction

Pension systems were originally designed to provide a guaranteed income. However, many pension providers have switched to defined contribution systems due to major problems with guaranteeing benefit levels. Related to this is a change in investment decisions that are not focused on retirement income, but on the value of the fund. Several authors (e.g. Merton (2014), Kobor and Muralidhar (2020), Merton and Muralidhar (2020)) point out that this information is useless for a future retiree. The relevant information is the replacement rate (ratio of last salary to pension). Common securities like stocks and bonds are too risky for this purpose. Muralidhar (2015), Muralidhar et al. (2016), Merton and Muralidhar (2017) propose new financial securities as suitable risk-free investment instruments: Standard-of-Living Forward-starting Income-only Securities (SelFIES). SelFIES has a defined stream of payments (e.g. \$1 per year) starting at retirement for a period of e.g. 20 years. Payments are indexed by consumption per capita from the time of issuance.

The real form of SeLFIES can be found, e.g., in Brazil. In 2022, the Brazilian Pension Bond NTN-B1 was officially issued (traded from January 30, 2023), which was subsequently renamed Tesouro RendA+, abbreviated RendA+. Unlike the original SelFIES proposal, indexation is based on inflation, which does not significantly change the basic characteristics of SeLFIES.

In Kobor and Muralidhar (2020), the authors presented GLIDeS: Goal-based, Lifetime Income-focused, Dynamic Strategy that allocates resources between equities and SelFIES. They showed that introducing a new risk-free asset in combination with this strategy leads to acceptable and relatively secure retirement income.

In this paper, we follow up on the results of Kilianová and Ševčovič (2013) and Černý and Melicherčík (2020), where the authors present a stochastic dynamic model with maximization of the expected utility of the final amount of savings. However, we replace the amount of savings with retirement income and the risk-free asset with SeLFIES. We show that our approach changes the traditional ranking of riskiness of the assets used.

The paper is structured as follows. In Section 1, we formulate a stochastic dynamic programming model with the maximization of expected utility of retirement income. In the next section, we calibrate the model and analyze the riskiness of the assets used. In Section 4, we present the model results and in the last section, we conlude.

1 Model formulation

In our calculations we use the following assets: Treasury securities with constant maturities, Treasury Inflation-Protected Securities (TIPS), Index S&P 500 (SPX), SeLFIES. We suppose, that SeLFIES is an annuity paying 1 real money unit yearly beginning from the time of retirement for 20 years.

Denote by M_i, $i = 1, 2, \ldots n$ the real prices of Treasury bonds, TIPS, SPX and by S the real price of SelFIES. Assume that the prices follow Geometric Brownian motion:

$$dM/M = \mu\, dt + \sigma\, dW_t\,, \tag{1}$$
$$dS/S = \mu_S\, dt + \sigma_{S*}\, dW_t \tag{2}$$

where W are $n+1$ uncorrelated Brownian motions, $\mu \in \mathbf{R}^n$, $\sigma \in \mathbf{R}^{n\times(n+1)}$ and $\sigma_{S*} \in \mathbf{R}^{1\times(n+1)}$. We suppose that the correlation matrices $\Sigma_M = \sigma\sigma^\mathsf{T}$ and $\Sigma_{MS} = (\sigma;\sigma_{S*})(\sigma;\sigma_{S*})^\mathsf{T}$ are regular.

SeLFIES can serve as a risk-free asset for pension saving. Let $\tilde{M}_i = M_i/S$. $\tilde{M}_i$ thus represents the real retirement income that one can buy with the current value of asset i. Using Itô's formula one can calculate:

$$d\tilde{M}/\tilde{M} = \tilde{\mu}\, dt + \tilde{\sigma}\, dW_t \tag{3}$$

where

$$\tilde{\mu}_i = \mu_i - \mu_S + \sigma_S^2 - \sigma_{i*}\sigma_S^\mathsf{T}\,, \tag{4}$$
$$\tilde{\sigma}_{i*} = \sigma_{i*} - \sigma_S \tag{5}$$

and σ_{i*} and $\tilde{\sigma}_{i*}$ denote the $i-$th row of the matrices σ and $\tilde{\sigma}$ respectively and $\sigma_S^2 = \sigma_{S*}\sigma_{S*}^\mathsf{T}$.

Denote by I_t the real income paid annually 20 years after retirement that can be purchased at time t from the assets owned. I_t then follows the following stochastic differential equation:

$$dI_t = y_t\, dt + \pi_t I_t\, d\tilde{M}/\tilde{M} \tag{6}$$

where π_t represent the proportions invested in particular assets and y_t is the contribution per unit time expressed in the amount of real retirement income. If the model allows investment directly in SeLFIES, then $\pi_t \mathbf{1} \leq 1$ and the weight $1 - \pi_t \mathbf{1}$ corresponds to the SeLFIES. Otherwise $\pi_t \mathbf{1} = 1$. The saver with retirement time T can then make decisions based on

$$\sup_{\pi_t} \mathbf{E}(U(I_T)) \tag{7}$$

with (6) and $\pi_t \mathbf{1} \leq (=)1$. It is known from the theory of stochastic dynamic programming that the so-called value function

$$V(x,t) := \sup_{\pi|_{(t,T)}} \mathbf{E}(U(I_T)|I_t = x) \tag{8}$$

subject to the terminal condition $V(x,T) = U(x)$, can be used for solving the stochastic dynamic optimization problem (7). The value function satisfies the following Hamilton-Jacobi-Bellman equation:

$$\frac{\partial V}{\partial t} + \sup_{\pi_t} \{ \frac{\partial V}{\partial x}(y_t + \pi_t \tilde{\mu} x) + \frac{x^2}{2} \frac{\partial^2 V}{\partial x^2} \pi_t \tilde{\Sigma} \pi_t^{\mathsf{T}} \} = 0 \tag{9}$$

where $\tilde{\Sigma} = \tilde{\sigma}\tilde{\sigma}^{\mathsf{T}}$.

2 Calibration

For the specific implementation of the model, we used the following assets:

- U.S. Treasury securities with fixed maturities (0.5, 1, 2, 3, 5, 7, 10, 20, 30 years). The corresponding yields (called Treasury Constant Maturity (CMT) rates) are calculated by the U.S. Treasury from the daily yield curve. We denote the government bond data as GS with the maturity added.
- S&P500 stock index
- Treasury inflation-indexed bonds (Treasury Inflation-Protected Securities - TIPS) with fixed maturities 5, 7, 10 and 20 years. These are government debt securities whose principal value is adjusted for inflation. Interest is paid on the adjusted principal, and at maturity, investors receive the adjusted principal, or the original principal, whichever is greater.
- SeLFIES (Standard of Living indexed, Forward-starting, Income-only Securities) with conversion periods 1-40 years used in different phases of retirement investing.

In order to calibrate the model, we calculated historical monthly real returns for the period from August 2004 to May 2025. The returns for Treasury securities and TIPS can be calculated directly from historical data provided by Federal

Asset	μ_i	$\mu_i^{(c)}$	σ_i
TIPS5	0.44	0.36	4.13
TIPS7	0.70	0.56	5.34
TIPS10	0.92	0.74	5.97
TIPS20	1.45	0.97	9.77
GS6M	0.85	0.85	0.61
GS1	0.87	0.87	0.79
GS2	0.97	0.96	1.36
GS3	1.12	1.09	2.05
GS5	1.49	1.43	3.45
GS7	1.84	1.73	4.69
GS10	2.21	2.01	6.19
GS20	3.03	2.54	9.94
GS30	3.45	2.68	12.39
SPX	6.65	5.53	14.95

Table 1: Parameters μ_i, $\mu_i^{(c)}$ and σ_i (in percentages) for corresponding assets.

Reserve Bank (FRED). The real returns of index S&P500 were calculated using historical data of the index (Investing.com) and CPI index from (FRED). Applying Itô's formula to (1) one has:

$$d \ln M = \mu^{(c)}\, dt + \sigma\, dW_t \tag{10}$$

where

$$\mu_i^{(c)} = \mu_i - \frac{1}{2}\sigma_i^2 \tag{11}$$

and $\sigma_i^2 = \sigma_{i*}\sigma_{i*}^{\mathsf{T}}$. From historical data we estimated vector $\mu^{(c)}$ and covariance matrix $\Sigma_M = \sigma\sigma^{\mathsf{T}}$. The values σ_i^2 are on the diagonal of the matrix Σ_M. The vector μ was calculated according to equation (11). Corresponding values μ_i, $\mu_i^{(c)}$ and σ_i for all assets are in Tab 1.

Using Svensson model, historical CMT data, and 30-year inflation estimates from the FRED, we calculated historical real yield curves. From these, we calculated SeLFIES values with a conversion period up to 40 years. We then calculated historical returns and estimated the parameters μ_S, $\mu_S^{(c)}$ and the covariance matrix Σ_{MS}. Values of parameters for selected conversion periods are in Tab. 2. One can observe extremely high volatilities for long conversion periods as a result of the high duration of the corresponding assets.

From the Σ_{MS} matrix, we calculated the σ and σ_{S*} matrices using the Cholesky decomposition. Denote by $\tilde{\sigma}_i$ the volatility of $\widetilde{M}$. Then one has:

$$\tilde{\sigma}_i^2 = (\sigma_{i*} - \sigma_{S*})(\sigma_{i*} - \sigma_{S*})^{\mathsf{T}} = \sigma_i^2 - 2\sigma_{i*}\sigma_{S*}^{\mathsf{T}} + \sigma_S^2\,. \tag{12}$$

The values of $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ calculated according to (4) and (12) for SeLFIES with selected conversion periods are in Tab 3.

Conversion	μ_S	$\mu_S^{(c)}$	σ_S
0	1.56	1.40	5.72
5	2.13	1.74	8.76
10	2.58	1.85	12,.5
15	3.32	1.93	16.64
20	5.22	2.07	25.10
25	10.31	2.27	40.10
30	22.65	2.51	63.47
35	49.38	2.79	96.54
40	101.73	3.09	140.46

Table 2: Parameters μ_C, $\mu_S^{(c)}$ and σ_S (in percentages) for SeLFIES with selected conversion periods.

Conversion	40		30		15		0	
Parameter	$\tilde{\mu}_i$	$\tilde{\sigma}_i$	$\tilde{\mu}_i$	$\tilde{\sigma}_i$	$\tilde{\mu}_i$	$\tilde{\sigma}_i$	$\tilde{\mu}_i$	$\tilde{\sigma}_i$
TIPS5	95.44	140.12	17.53	62.74	-0.42	15.24	-0.90	5.31
TIPS7	95.63	140.12	17.60	62.52	-0.33	14.48	-0.70	5.29
TIPS10	96.06	140.29	17.86	62.66	-0.17	14.27	-0.53	5.01
TIPS20	96.12	140.17	17.76	62.14	-0.16	12.74	-0.18	7.05
GS6M	96.39	140.45	18.47	63.46	0.29	16.62	-0.39	5.67
GS1	96.37	140.42	18.46	63.40	0.29	16.50	-0.38	5.52
GS2	96.39	140.37	18.47	63.28	0.33	16.20	-0.31	5.17
GS3	96.50	140.35	18.53	63.16	0.41	15.81	-0.19	4.78
GS5	96.77	140.30	18.67	62.85	0.60	14.81	0.11	3.97
GS7	97.00	140.25	18.79	62.57	0.75	13.85	0.39	3.51
GS10	97.04	140.08	18.80	62.13	0.87	12.59	0.68	3.55
GS20	97.13	139.77	18.79	61.25	1.10	9.93	1.32	5.94
GS30	96.53	139.24	18.46	60.49	1.13	8.72	1.62	8.23
SPX	102.94	141.77	24.78	65.96	6.24	23.00	5.45	16.25

Table 3: Parameters $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ (in percentages) calculated according to (4) and (12) for SeLFIES with selected conversion periods.

Based on Tab. 3, it can be concluded that TIPS are not proper investment assets. Treasury securities with the same maturity have higher $\tilde{\mu}_i$ values and comparable or lower volatilities $\tilde{\sigma}_i$. For conversion periods except for immediate conversion, decreasing $\tilde{\sigma}_i$ volatility can be observed with increasing maturity. This is consistent with equation (12), where the term $\sigma_{i*}\sigma_{S*}^{\intercal}$ represents the covariance of the $i-$th asset return and SeLFIES. This covariance is naturally higher (for longer conversion periods) for Treasury securities with higher maturities. The aforementioned downward trend is broken for Treasury securities with maturities of 10, 20, 30, where the duration exceeds the SELFIE duration. On the other hand, the slightly increased volatility of $\tilde{\sigma}_i$ is compensated by the increase in $\tilde{\mu}_i$. As for the values belonging to the SPX index, we can observe significantly higher $\tilde{\mu}_i$ values along with higher $\tilde{\sigma}_i$ values. Overall, comparing Tab 1 and Tab 3, we can conclude that by reversing the thinking from value-oriented to income-oriented, there is a big change in the risk perspective. Treasury securities with low maturities are riskier for an income-oriented approach than those with higher maturities. From an income target perspective, this makes sense because they have a higher correlation with SeLFIES, which represent future income.

3 Results

For the analysis, we used standard utility functions of the form

$$U_\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}\,,\ \gamma > 0\,,\ \gamma \neq 1\,. \tag{13}$$

Tha analysis can be extended to $\gamma = 1$ with $U_1(x) = \ln(x)$. Using the method presented in Kilianová and Ševčovič (2013), we calculated the solution to equation (9) along with the optimal investment strategy $\pi_t = \pi(x,t)$. We report the utility of optimal strategies corresponding to problem (7) in terms of certainty equivalent wealth (CE) and certainty equivalent internal rate of return (IRR). The certainty equivalent wealth is calculated from the formula

$$U(CE) = \sup_{\pi_t} \mathbf{E}(U(I_T)) = V(x,0)\,. \tag{14}$$

The certainty equivalent internal rate of return is given as the interest rate ρ satisfying

$$CE = \int_0^T e^{\rho(T-t)} y_t\, dt\,. \tag{15}$$

To begin, let us assume a unit annual wage with zero real growth and a contribution of $y_t = 1.25\%$, which corresponds to a replacement rate of 50% assuming 40 years of savings. We consider 3 different levels of risk aversion $\gamma = 2$, 5, and 8. In the case of using SeLFIES, we assume conversion at the time of retirement. Values of CE and IRR for the case with (+S) and without using SeLFIES (-S) can be found in Tab. 4. One can observe that a 50% replacement rate was achieved in all cases.

Gamma	CE (+S)	IRR (+S)	CE (-S)	IRR (-S)
8	0.7093	1.66%	0.5034	0.03%
5	0.8471	2.44%	0.6303	1.12%
2	1.4326	4.58%	1.1781	3.81%

Table 4: Values of CE and IRR for the case with (+S) and without using SeLFIES (-S).

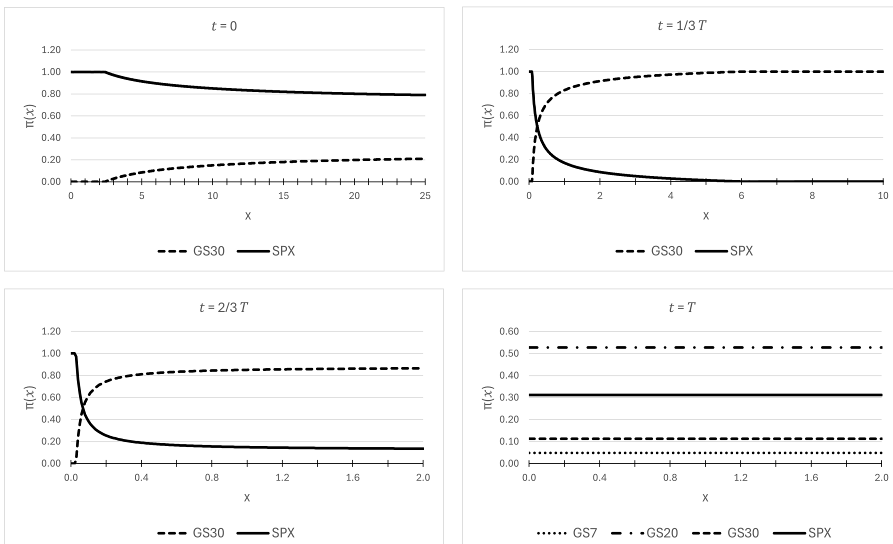

Figure 1: Optimal asset allocations for $\gamma = 8$ and selected time periods without using SeLFIES.

Optimal asset allocations for $\gamma = 8$ and selected time periods without using SeLFIES are shown in Fig. 1. Those with using SeLFIES are in Fig. 2. In line with the results of Cairns et al. (2006) and Černý and Melicherčík (2020) for so-called stochastic life cycle investing with credit constraints, in both cases, for a small level of accumulated capital, we see investments in the riskiest assets with the highest value of $\tilde{\mu}_i$ (SPX). Subsequently, investments are gradually shifted to less risky assets (GS30). In the final phase of savings, the duration of GS30 is higher than the duration of SeLFIES, and for this reason, investments are transferred to GS20. In the case of investing without the SeLFIES option, investments in GS7 also appear. In the case with using SeLFIES, 16% of investments are in this security in the final stage.

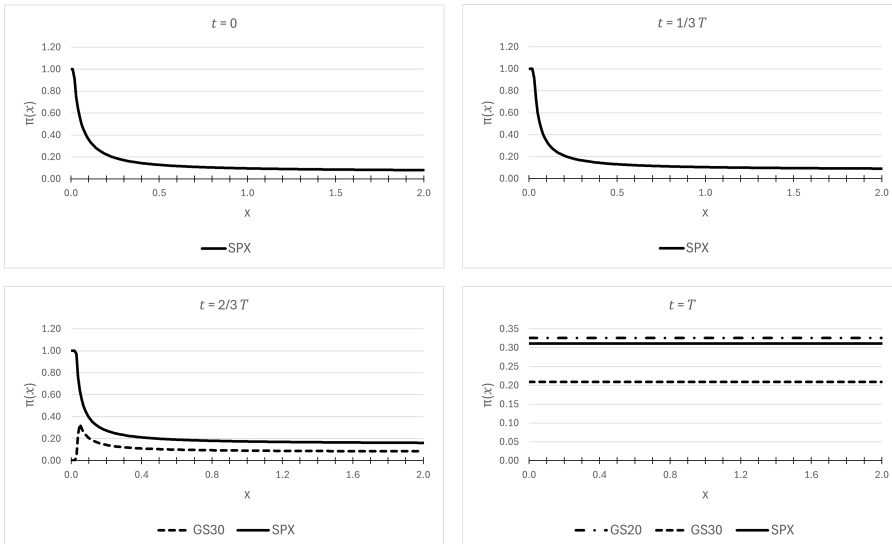

Figure 2: Optimal asset allocations for $\gamma = 8$ and selected time periods with using SeLFIES.

4 Conclusions

Unlike classic retirement investment models, which track risk and the value of pension savings, in this paper we optimize retirement income. We have formulated an income-oriented retirement investment model where the expected utility of retirement income is maximized. The investment strategy is the optimal feedback of a stochastic dynamic problem. With the paradigm shift from value-oriented to income-oriented, there is also a change in the risk ranking of assets. Government bonds with longer maturities are less risky assets compared to those with shorter maturities. This is because bonds with higher maturities have a higher correlation with SeLFIES. In line with the results of Cairns et al. (2006) and Černý and Melicherčík (2020), in the early stages of retirement savings it is optimal to invest in stocks and later shift investments to less risky assets, which in the case of the presented model are government bonds with higher maturities. We present the optimal level of savings in the form of a certainty equivalent and an internal rate of return of the certainty equivalent. Optimal strategies yield satisfactory levels of replacement rates.

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